SELECTION PROBLEMS '26

Below you will find some problems of various difficulty, some of which are open ended[1]. It is certainly *not* necessary to solve all the problems in order to be selected to participate in the Apex Math summer program ! We are more interested in seeing your unique way of approaching the problem and in the methods which you come up with than in seeing a full solution. We encourage you to write up (and develop to the extent that you feel inspired) any idea that you have, even if it only concerns a special case or if it does not lead to a complete solution. Your curiosity in exploring the patterns that you notice will be much valued.

PROBLEM 1.

Let m be a fixed positive integer. What can you say about the sequence

$$1^1, 2^2, 3^3, \dots$$

modulo m (hint: could it be periodic ? When it is, what is the period ?) ? How about the sequence

$$1^{1^1}, 2^{2^2}, 3^{3^3}, \dots ?$$

And now what do you think of the sequence

$$1^1, 1^1 + 2^2, 1^1 + 2^2 + 3^3, \dots$$

(still modulo m) ? These numbers can get BIG so do not hesitate to use a computer program to experiment!

PROBLEM 2.

Can you (or not ?) find a polynomial $P(x)$ of a real variable such that the restriction $P_{|\mathbb{Q}}$ of P to $\mathbb{Q}$ is injective but P is not injective (on $\mathbb{R}$) ? Can we replace *injective* by *surjective* ? By *bijective*? Anything interesting if we consider polynomials of several variables ?

PROBLEM 3.

(i) For $n > 1$ we let $z(n)$ be the number of zeroes in the digital expansion of n. For which real numbers a does the sum

$$\sum_{n \geq 1} \frac{a^{z(n)}}{n^2}$$

converge (and what does this mean ?) ?

[1] Open-ended problems are those which have many solutions or no solutions as defined; they challenge us to ask side questions, to seek better questions and to consider many different angles, each shedding light on a different aspect of the problem studied.

(ii) Let $a_0, a_1, \dots$ be positive real numbers such that $\lim_{n\to\infty} a_n = 0$ (what does this mean ?). Can you construct a sequence of positive real numbers $b_0, b_1, \dots$ such that

$$\sum_n b_n = \infty, \qquad \text{but} \qquad \sum_n a_n b_n < +\infty \quad ?$$

PROBLEM 4.

On the planet $\mathcal{A}p_X$ (whose inhabitants are called $\mathcal{A}$pes), there are two kinds of animals: the $\mathcal{A}pp$s, who have two antennas each and the $\mathcal{A}ppp$s, who have three antennas each. When a cosmic ray hits $\mathcal{A}p_X$, it is automatically captured by an antenna (with equal probability), with the following effects:

(i) If the antenna belongs to an $\mathcal{A}pp$ then that $\mathcal{A}pp$ grows an extra antenna and mutates into an $\mathcal{A}ppp$;

(ii) If the antenna belongs to an $\mathcal{A}ppp$ then that $\mathcal{A}ppp$ grows an extra antenna and splits into two $\mathcal{A}pp$s.

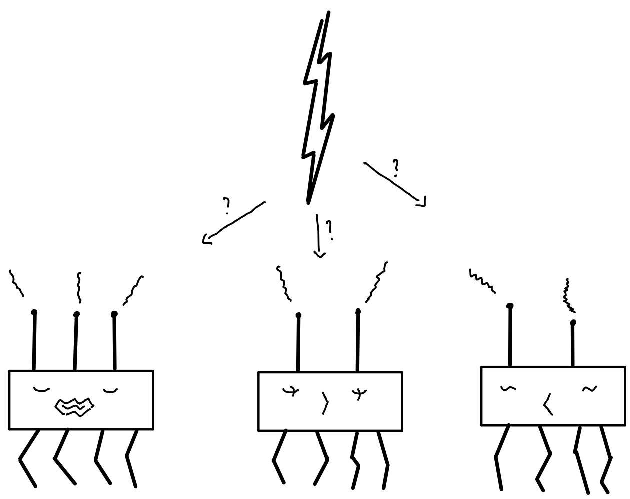

Figure 1: A herd of peaceful animals from planet $\mathcal{A}p_X$ about to be hit by a cosmic ray.

Assume that we start with a population of x $\mathcal{A}pp$s and y $\mathcal{A}ppp$s. What can you say about the expected distribution of animals on $\mathcal{A}p_X$ after c cosmic rays have entered the atmosphere ? In how many ways can you generalize this problem ?

PROBLEM 5.

Given a collection X of subsets of a given set S, we denote by $I(X)$ the set of all subsets $T \subset S$ which intersect all elements of X. Does $I(X)$ always have a minimal element (for the inclusion order) ?

PROBLEM 6.

A subset $S \subset \mathbb{R}^2$ is said to be *dense* if every disk (with nonzero radius!) in $\mathbb{R}^2$ intersects S. Do you think that it is possible to find a partition of $\mathbb{R}^2$ into infinitely many dense pieces such that any straight line intersects infinitely many of these sets ? Every line passing through a point (x, y) with rational coordinates ? Through two distinct points $(x, y), (w, z)$ with rational coordinates ? Can you find a nontrivial finite partition for which every line as above intersects *all* the pieces ? Do you dare to think about the higher-dimensional problem ?

PROBLEM 7.

A lattice point $(p, q) \in \mathrm{Z}^2$ is *primitive* if $gcd(p, q) = 1$. What is the maximal R radius of a disk $D = D((x, y); R)$ which avoids all primitive lattice points (if there is any :)) ? Can you formulate a 3d version ? A 4d on e? How do things change if I allow *one* primitive point ?